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On the existence of electromagnetic-hydrodynamic waves.

By

HANNES ALFVÉN.

With 2 figures in the text.

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§ 1. If a conducting liquid is placed in a constant magnetic field, a mechanical motion in the liquid will in general give rise to an e. m. f., which produces electric currents. The interaction between the magnetic field and these currents causes mechanical forces which change the state of motion of the liquid.

Thus the application of a magnetic field to a conducting liquid causes a mutual interaction between hydrodynamic motion and electric current. Thus kinetic energy can be converted into electromagnetic energy and *vice versa*. This mechanism makes possible the existence of a kind of combined *electromagnetic-hydrodynamic wave*, which — as far as I know — has as yet attracted no attention.

For the electromagnetic vectors we have

$$\operatorname{rot} H = \frac{4\pi}{c} i \tag{1}$$

$$\operatorname{rot} E = -\frac{1}{c}\frac{\partial B}{\partial t} \tag{2}$$

$$B = \mu H \tag{3}$$

$$i = \sigma \left(E + \frac{v}{c} \times B\right) \tag{4}$$

where E is the electric and H the magnetic field, i the electric current density, v the velocity of the liquid, σ the electric conductivity, μ the permeability, and c the velocity of light.

These equations must be combined with the hydrodynamic equation

$$\vartheta \frac{dv}{dt} = \frac{1}{c}(i \times B) - \text{grad } p \tag{5}$$

where ϑ denotes the mass density and p the hydrostatic pressure. If we suppose that the liquid is uncompressible, we have

$$\text{div } v = 0. \tag{6}$$

§ 2. In order to study the phenomenon under as simple conditions as possible, let us suppose that the primary magnetic field H_0 is homogeneous and parallel to the z-axis of an orthogonal coordinate system, the conductivity σ is infinite, and ϑ is constant.

The magnetic field consists of the primary field H_0 and the field H' which is caused by the current i. In order to study a *plane wave* in the direction of H_0, we assume that all vectors are independent of x and y (but depend upon z and the time t).

This implies that according to (1) and (2) we have $i_z = 0$ and $H_z = \text{const} = H_0$. Further, according to (6) we may put $v_z = 0$.

If we turn the coordinate system in such a way that $i_y = 0$, we obtain from (1)

$$\begin{aligned} i_x &= -\frac{c}{4\pi}\frac{\partial H_y}{\partial z} \\ i_y &= i_z = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} H_x &= \text{const} = 0 \\ H_z &= H_0 . \end{aligned} \tag{8}$$

We introduce these values into (5). As according to our assumptions grad p can have no components perpendicular to the z-axis, we obtain

$$\frac{\partial v_x}{\partial t} = 0; \quad v_x = \text{const} = 0$$

$$\frac{\partial v_y}{\partial t} = \frac{\mu H_0}{4\pi\vartheta} \frac{d H_y}{dz} \tag{9}$$

$$v_z = 0$$

and further

$$\frac{d p}{d z} = -\frac{\mu}{8\pi} \frac{d (H_y^2)}{d z}. \tag{10}$$

Because i is finite, equation (4) gives

$$E = -\mu \frac{v}{c} \times H$$

or with (8) and (9)

$$E_x = -\mu \frac{v_y}{c} H_0$$

$$E_y = E_z = 0. \tag{11}$$

Equation (2) gives

$$\mu \frac{d H_y}{d t} = -c \frac{d E_x}{d z}. \tag{12}$$

Combining (12), (11) and (9) we obtain

$$\frac{d^2 H_y}{d t^2} = \frac{\mu H_0^2}{4\pi\vartheta} \frac{d^2 H_y}{d z^2} \tag{13}$$

which means a wave in the direction of the z-axis with the velocity

$$V = \frac{H_0 \sqrt{\mu}}{\sqrt{4\pi\vartheta}}. \tag{14}$$

The velocity of the electromagnetic-hydrodynamic wave is independent of the frequency as well as of the amplitude.

§ 3. If we put

$$H_y = A \sin \omega \left(t - \frac{z}{V}\right) \tag{15}$$

we find

$$v_y = - \frac{A \sqrt{\mu}}{\sqrt{4 \pi \vartheta}} \sin \omega \left(t - \frac{z}{V}\right) \tag{16}$$

$$i_x = A \frac{c\,\omega}{H_0} \sqrt{\frac{\vartheta}{4 \pi \mu}} \cos \omega \left(t - \frac{z}{V}\right) \tag{17}$$

$$E_x = A \frac{H_0 \sqrt{\mu^3}}{c \sqrt{4 \pi \vartheta}} \sin \omega \left(t - \frac{z}{V}\right) \tag{18}$$

$$p = p_0 - \frac{\mu A^2}{8 \pi} \sin^2 \omega \left(t - \frac{z}{V}\right). \tag{19}$$

The magnetic lines of force, which with no waves were straight lines

$$x = x_0, \quad y = y_0 \tag{20}$$

change their shape into sine curves:

$$x = x_0$$

$$y = y_0 + \frac{A \sqrt{\mu}}{\omega \sqrt{4 \pi \vartheta}} \cos \omega \left(t - \frac{z}{V}\right). \tag{21}$$

Differentiating (21) we find that the magnetic lines of force oscillate with the same velocity (given by (16)) as the liquid.

§ 4. The phenomenon can also be treated along quite different lines. Suppose that we have a homogeneous magnetic field in a perfectly conducting liquid. The magnetic lines of force can be considered as elastic strings according to the usual mechanical picture of electrodynamical phenomena. In view of the infinite conductivity, every motion (perpendicular to the field) of the liquid in relation to the lines of force is forbidden because it would give infinite eddy currents. Thus the matter of the liquid is »fastened» to the lines of force, constituting a series of strings.

A swinging string obeys the equation

$$m \frac{d^2 y}{d t^2} = S \frac{d^2 y}{d z^2} \tag{22}$$

if the string has the direction of the z-axis and swings parallel to the y-xis. S means the tension and m the mass per unit length.

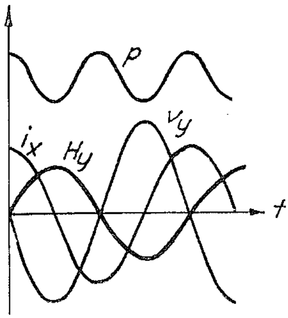

Fig. 1. Current i_x, velocity v_y, variable component of magnetic field H_y, and pressure p, as functions of the time t.

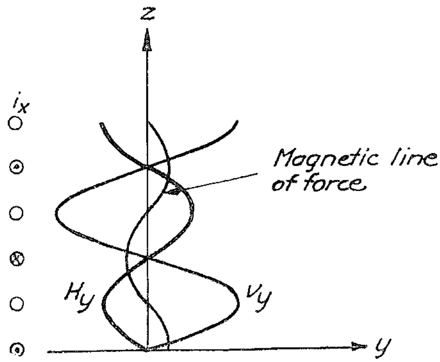

Fig. 2. Velocity v_y and variable component of magnetic field H_y as functions of z. Geometrical form of a magnetic line of force. To the left is shown where i_x is zero (○), directed upwards (⊙) or downwards (⊕) through the paper.

In such a string the velocity of transverse waves is

$$V = \sqrt{\frac{S}{m}}. \tag{23}$$

In our case we substitute for the string a filament of the liquid with unit cross-section. Thus m corresponds to ϑ. If the string in a certain moment has the shape

$$y = f(z) \tag{24}$$

its length has increased by

$$\int \left[\sqrt{1 + \left(\frac{d y}{d z}\right)^2} - 1 \right] d z,$$

which means that its potential energy is

$$W = S\int\left[\sqrt{1+\left(\frac{dy}{dz}\right)^2}-1\right]dz \approx \frac{S}{2}\int\left(\frac{dy}{dz}\right)^2 dz \tag{25}$$

The magnetic lines of force have the same shape as the string if, to the original field H_0 in the z-direction, we add

$$H_y = H_0 \frac{dy}{dz}. \tag{26}$$

When doing so we increase the energy by the amount

$$W = \frac{\mu}{8\pi}\int H_y^2\, dz = \frac{\mu H_0^2}{8\pi}\int\left(\frac{dy}{dz}\right)^2 dz. \tag{27}$$

The energies (25) and (27) become equal if we put

$$S = \frac{\mu H_0^2}{4\pi}. \tag{28}$$

Introducing this expression in (23) we obtain

$$V = \frac{H_0\sqrt{\mu}}{\sqrt{4\pi\vartheta}}.$$

which is in accordance with (14).

§ 5. Electromagnetic-hydrodynamic waves are probably very important in solar physics. The sun's general magnetic field constitutes the primary field H_0 in which the waves move. Owing to its ionization, solar matter is a good electrical conductor. The fact that mechanical motions as well as strong magnetic fields are observed in sunspots indicate that they may be associated with waves of this type, although more complicated than the plane waves we have studied.

During the 11-year period the sunspot zone moves from a latitude of about 30° towards the equator with a velocity of the order of 100 cm sec^{-1}. As the general magnetic field is of the order of $H_0 = 15$ gauss, the velocity of an electromagnetic-hydrodynamic wave amounts to 100 cm sec^{-1} if $\mu = 1$ and

$$\vartheta = \frac{H_0^2}{4\pi V^2} = 0.002 \text{ g cm}^{-3}.$$

The solar density has this value at about one tenth of the solar radius below the surface. The original cause of the sunspots may very well be situated at that depth. Thus the cause of sunspots may be electromagnetic-hydrodynamic waves probably originating in the sun's interior and reaching the surface in the sunspot zones.

The problem of electromagnetic-hydrodynamic waves in the sun will be treated in a later publication.

As the term »electromagnetic-hydrodynamic waves» is somewhat complicated, it may be convenient to call the phenomenon »*magneto-hydrodynamic*» waves. (The term »hydromagnetic» is still shorter but not quite adequate.)

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